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Löwner inequality of indefinite type

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Dedicated to Professor Peter Lancaster on his 75th birthday

Abstract

A selfadjoint involutive matrix J provides $\mathbf{C}^n$ with an (indefinite) inner product $[x, y] \equiv \langle Jx, y\rangle$. For a pair of J-selfadjoint matrices A, B, the J-order relation $A \overset{J}{\geqslant} B$ is defined as $[Ax, x] \geqslant [Bx, x]$ for all x.

We will show that if A, B are J-selfadjoint matrices such that all eigenvalues of A, B are real and contained in an interval (α, β) then, for any operator monotone function $f(t)$ on (α, β), the matrices $f(A), f(B)$ are well defined by the Riesz–Dunford integral and

$$A \overset{J}{\geqslant} B \Longrightarrow f(A) \overset{J}{\geqslant} f(B).$$

When $J = I$ and $f(t) = t^{\frac{1}{2}}$ on $(0, \infty)$, this is the classical Löwner inequality.

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1. Introduction

It is a well-known theorem of Löwner that for a pair of selfadjoint $n \times n$ matrices A, B

$$A \geqslant B \geqslant 0 \Longrightarrow A^{\frac{1}{2}} \geqslant B^{\frac{1}{2}},$$

where order relation $A \geqslant B$ means that $A - B$ is positive semi-definite, or equivalently

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$$\langle Ax, x\rangle \geqslant \langle Bx, x\rangle \quad (x \in \mathbf{C}^n).$$

($A > 0$ means that A is positive definite, that is, $A \geqslant 0$ and A is invertible.)

A real valued continuus function $f(t)$ defined on a (finite or infinite) interval (α, β) is said to be *matrix monotone* of order n on (α, β) if for any pair of $n \times n$ selfadjoint matrices A, B with $\sigma(A), \sigma(B) \subset (\alpha, \beta)$ ($\sigma(A)$ denoting the set of eigenvalues of A)

$$A \geqslant B \Longrightarrow f(A) \geqslant f(B),$$

where $f(A)$ is defined by the usual functional caluculus for a selfadjoint matrix.

Further $f(t)$ is said to be *operator monotone* on (α, β) if it is matrix monotone of all order on (α, β). The naming "*operator monotone*" comes from the fact that, for a function matrix monotone of all order, the order preserving relation can be automatically extended to the case of a pair of selfadjoint operators on a Hilbert space. The above mentioned Löwner theorem says that the square root function $f(t) = t^{\frac{1}{2}}$ is operator monotone on $(0, \infty)$.

One of many deep results of Löwner (see [4] as a modern version) concerning characterizations of operator monotone functions on (α, β) is that $f(t)$ admits analytic continuation $f(\zeta)$ to the domain $\mathbf{C} \setminus \{(\infty, \alpha] \cup [\beta, \infty)\}$ such that

$$\mathrm{Im}(f(\zeta)) \cdot \mathrm{Im}(\zeta) > 0 \quad (\zeta \in \mathbf{C}, \mathrm{Im}(\zeta) \neq 0).$$

Given a *selfadjoint involution* J, that is, $J = J^*$, $J^2 = I$, let us consider an (*indefinite*) *inner product* $[\cdot, \cdot]$ induced by J;

$$[x, y] \equiv \langle Jx, y\rangle \quad (x, y \in \mathbf{C}^n).$$

For a matrix A, its J-*adjoint* $A^\#$ is defined naturally by

$$[Ax, y] = [x, A^\# y] \quad (x, y \in \mathbf{C}^n),$$

which is equivalent to say that

$$A^\# \equiv JA^*J.$$

A matrix A is said to be J-*selfadjoint* if $A = A^\#$ or equivalently JA is selfadjoint, that is,

$$JA = A^*J.$$

For a pair of J-selfajoint matrices A, B, let us define an order relation $A \overset{J}{\geqslant} B$ as

$$[Ax, x] \geqslant [Bx, x] \quad (x \in \mathbf{C}^n),$$

or equivalently $JA \geqslant JB$.

J-selfadjointness of a matrix A does not always imply that all eigenvalues of A are real. If A is J-selfadjoint and $I \overset{J}{\geqslant} A$ then all eigenvalues of A are real. For, in this case, $I - A$ is a prodcut of selfadjoint J and a positive semi-definite matrix.

Recall that a matrix A is called a J-*contraction* if $I \overset{J}{\geqslant} A^\# A$, that is, $J \geqslant A^*JA$ or equivalently

$$[x, x] \geqslant [Ax, Ax] \quad (x \in \mathbf{C}^n).$$

As mensiond above, for a J-contraction A all eigenvalues of the product $A^{\#}A$ are real. Further more it is known as a result of Potapov–Ginzburg (see [1, Chapter 2, Section 4]) that in this case all eigenvalues are non-negative real.

If all eigenvalues of a J-selfadjoint matrix A are real and $\sigma(A) \subset (\alpha, \beta)$, for any operator monotone function $f(t)$ on (α, β) we can define $f(A)$ by the Riesz–Dunford integral

$$f(A) = \frac{1}{2\pi i} \int_C f(\zeta)(\zeta I - A)^{-1}\, \mathrm{d}\zeta,$$

where C is a closed rectifiable contour in the domain of analytic continuation of $f(t)$, surrounding $\sigma(T)$ positively in its interior. $f(A)$ becomes J-selfadjoint.

Of course, when A is selfadjoint, this integral produces the same matrix as that defined by the usual functional caluculus for a selfadjoint matrix.

Our main result (Theorem 4) is that if all eigenvalues of two J-selfadjoint matrices A, B are real and $\sigma(A), \sigma(B) \subset (\alpha, \beta)$ then

$$A \overset{J}{\geqslant} B \Longrightarrow f(A) \overset{J}{\geqslant} f(B)$$

for any operator monotone function $f(t)$ on (α, β).

For a J-contraction A, even when $A^{\#}A$ has 0 as its eigenvalue, we can define its square-root $(A^{\#}A)^{\frac{1}{2}}$ by the operator monotone function $f(t) \equiv t^{\frac{1}{2}}$ on $(0, \infty)$, called the *J-modulus* of A. A consequence (Corollary 7) of our theorem is that

$$I \overset{J}{\geqslant} A^{\#}A \overset{J}{\geqslant} B^{\#}B \Longrightarrow I \overset{J}{\geqslant} (A^{\#}A)^{\frac{1}{2}} \overset{J}{\geqslant} (B^{\#}B)^{\frac{1}{2}}.$$

2. Inequalities

The *inertia* of a matrix A is a triple of non-negative integers $(\pi_-(A), \pi_0(A), \pi_+(A))$ where $\pi_+(A)$ (resp. $\pi_-(A)$) is the number of eigenvalues of A with positive (resp. negative) real part (with multiplicities counted) while $\pi_0(A)$ is the number of eigenvalues on the imaginary axis.

It is evident that the inertia is invariant for similarity. If A is selfadjoint and S is invertible, A and S^*AS have same inertia, and conversely if two invertible selfadjoint matrices A and B have same inertia then there is an invertible matrix S such that $A = S^*BS$.

The following lemma is known as an *inertia theorem* (see [6, Chapter 13, Section 1]).

Lemma 1. *For a matrix A, the condition $\pi_0(A) = 0$ is equivalent to the existence of an invertible selfadjoint matrix H such that*

$$HA + A^*H > 0.$$

In this inequality, A and the selfadjoint matrix H have necessarily same inertia.

Lemma 2. *Let A, B be invertible selfadjoint matrices. If all eigenvalues of the product AB are positive real, then A and B have same inertia.*

Proof. Since by assumption

$$\pi_-(AB) = \pi_0(AB) = 0,$$

by Lemma 1 there is positive definite H such that

$$H \cdot (AB) + (BA) \cdot H > 0.$$

Multiplying this inequality by $H^{-\frac{1}{2}}$ from both sides, we can see

$$\left(H^{\frac{1}{2}}AH^{\frac{1}{2}}\right) \cdot \left(H^{-\frac{1}{2}}BH^{-\frac{1}{2}}\right) + \left(H^{-\frac{1}{2}}BH^{-\frac{1}{2}}\right) \cdot \left(H^{\frac{1}{2}}AH^{\frac{1}{2}}\right) > 0,$$

which implies, again by Lemma 1, that $H^{\frac{1}{2}}AH^{\frac{1}{2}}$ and $H^{-\frac{1}{2}}BH^{-\frac{1}{2}}$ have same inertia, so that A and B have same inertia. This completes a proof. □

The following is due to Smul'yan [7] (see also [2,5]).

Lemma 3. *Let A, B be invertible selfadjoint matrices with same inertia. Then $A \geqslant B$ implies $A^{-1} \leqslant B^{-1}$.*

Proof. Write

$$A = |A|^{\frac{1}{2}}J_1|A|^{\frac{1}{2}} \quad \text{and} \quad B = |B|^{\frac{1}{2}}J_2|B|^{\frac{1}{2}},$$

where J_1 and J_2 are selfadjoint involutions. Since $|A|^{\frac{1}{2}}$ and $|B|^{\frac{1}{2}}$ are invertible selfadjoint, A and J_1 have same inertia and B and J_2 have same inertia. Therefore by assumption J_1 and J_2 have same inertia. Then there is an invertible matrix S such that $J_2 = S^*J_1S$ (and hence also $J_2 = S^{-1}J_1(S^*)^{-1}$).

Now $A \geqslant B$ implies

$$J_1 \geqslant |A|^{-\frac{1}{2}}|B|^{\frac{1}{2}}S^*J_1S|B|^{\frac{1}{2}}|A|^{-\frac{1}{2}}.$$

This means that $S|B|^{\frac{1}{2}}|A|^{-\frac{1}{2}}$ is a J_1-contraction. Then according to a theorem of Potapov–Ginzburg (see [1, Chapter 2, Section 4]) its adjoint $|A|^{-\frac{1}{2}}|B|^{\frac{1}{2}}S^*$ is again a J_1-contraction, that is,

$$J_1 \geqslant S|B|^{\frac{1}{2}}|A|^{-\frac{1}{2}}J_1|A|^{-\frac{1}{2}}|B|^{\frac{1}{2}}S^*$$

or

$$|B|^{-\frac{1}{2}}S^{-1}J_1(S^*)^{-1}|B|^{-\frac{1}{2}} \geqslant |A|^{-\frac{1}{2}}J_1|A|^{-\frac{1}{2}},$$

which is equivalent to $B^{-1} \geqslant A^{-1}$. This completes a proof. □

Now we are in position to prove our main result.

Theorem 4. *Let J be a selfadjoint involution, and A, B J-selfadjoint matrices with $\sigma(A), \sigma(B) \subset (\alpha, \beta)$. Then*

$$A \overset{J}{\geqslant} B \Longrightarrow f(A) \overset{J}{\geqslant} f(B)$$

for any operator monotone function $f(t)$ on (α, β).

Proof. Since $\sigma(A), \sigma(B)$ are bounded sets, we may assume that (α, β) is a finite interval.

Given an operator monotone function $f(t)$ on (α, β), it is easy to see that the function $g(t)$ defined by

$$g(t) \equiv f\Big(\frac{\beta - \alpha}{2} t + \frac{\alpha + \beta}{2}\Big)$$

is operator monotone on $(-1, 1)$, and for any matrix T with $\sigma(T) \subset (\alpha, \beta)$

$$f(T) = g\Big(\frac{2}{\beta - \alpha}\Big(T - \frac{\alpha + \beta}{2} I\Big)\Big).$$

Since $A \overset{J}{\geqslant} B$ implies

$$\frac{2}{\beta - \alpha}\Big(A - \frac{\alpha + \beta}{2} \cdot I\Big) \overset{J}{\geqslant} \frac{2}{\beta - \alpha}\Big(B - \frac{\alpha + \beta}{2} \cdot I\Big),$$

we may further assume that $(\alpha, \beta) = (-1, 1)$.

It is a version of the Löwner theorem, due to Bendat–Sherman [3], that any operator monotone function $f(t)$ on $(-1, 1)$ admits a representation

$$f(t) = f(0) + \int_{-1}^{1} \frac{t}{1 - t\lambda}\, \mathrm{d}m(\lambda)$$

where $\mathrm{d}m(\cdot)$ is a positive measure on $[-1, 1]$, and for any matrix T with $\sigma(T) \subset (-1, 1)$

$$f(T) = f(0) \cdot I + \int_{-1}^{1} T(I - \lambda T)^{-1}\, \mathrm{d}m(\lambda),$$

where the left hand side is defined by the Riesz–Dunford integral.

Therefore, for the assertion of Theorem 4, it suffices to prove that

$$JA \geqslant JB \quad \text{and} \quad \sigma(A), \sigma(B) \subset (-1, 1) \tag{$\dagger$}$$

implies

$$JA(I - \lambda A)^{-1} \geqslant JB(I - \lambda B)^{-1} \quad (-1 < \lambda < 1),$$

or equivalently

$$\frac{1}{\lambda} J(I - \lambda A)^{-1} \geqslant \frac{1}{\lambda} J(I - \lambda B)^{-1} \quad (0 < |\lambda| < 1). \tag{$\ddagger$}$$

Since $\sigma(A), \sigma(B) \subset (-1, 1)$ implies

$$\sigma(J(J-\lambda JA)) = \sigma(I-\lambda A) \subset (0,2) \quad (-1<\lambda<1)$$

and

$$\sigma(J(J-\lambda JB)) = \sigma(I-\lambda B) \subset (0,2) \quad (-1<\lambda<1),$$

it follows from Lemma 2 that J, $J-\lambda JA$, and $J-\lambda JB$ have same inertia.

Since, for $0<\lambda<1$, (†) implies

$$J(I-\lambda A) = J-\lambda JA \leqslant J-\lambda JB = J(I-\lambda B),$$

it follows from Lemma 3 that

$$(J-\lambda JA)^{-1} \geqslant (J-\lambda JB)^{-1},$$

which leads to

$$\frac{1}{\lambda}J(I-\lambda A)^{-1} \geqslant \frac{1}{\lambda}J(I-\lambda B)^{-1}.$$

In a similar way, when $-1<\lambda<0$ we have

$$(J-\lambda JA)^{-1} \leqslant (J-\lambda JB)^{-1},$$

so that

$$\frac{1}{\lambda}J(I-\lambda A)^{-1} \geqslant \frac{1}{\lambda}J(I-\lambda B)^{-1}.$$

These complete a proof of (‡). □

3. *J*-modulus

For $A>0$, the matrix defined by

$$\frac{1}{\pi}\int_0^1 A\{\lambda A+(1-\lambda)I\}^{-1}(\lambda(1-\lambda))^{-\frac{1}{2}}\,\mathrm{d}\lambda \tag{$*$}$$

coincides with the square root of A defined by the Riesz–Dunford integral. Even in the case $A \geqslant 0$ the integral $(*)$ is convergent and coincides with the (unique) positive semi-definite square roof of A.

If A is J-selfadjoint with $\sigma(A) \subset (0,\infty)$, the integral $(*)$ is convergent and coincides with the J-selfadjoint square root defined by the Riesz–Dunford integral.

When A is J-selfadjoint with $\sigma(A) \subset [0,\infty)$, we have to impose some condition for the convergence of the integral $(*)$.

Lemma 5. *Suppose that A is J-selfadjoint with $\sigma(A) \subset [0,\infty)$. If $I \overset{J}{\geqslant} A$, that is, $J \geqslant JA$, then the integral $(*)$ is convergent and gives a J-selfadjoint square root of A with non-negative eigenvalues.*

Proof. We may assume $\mathscr{M} \equiv \ker(A) \neq \{0\}$. Let $C \equiv J(I - A)$. Then it follows from $C \geqslant 0$ that

$$||C|| \cdot \langle Cx, x\rangle \geqslant ||Cx|| \quad (x \in \mathbf{C}^n).$$

This implies that

$$||C|| \cdot [y, y] \geqslant ||y||^2 \quad (y \in \mathscr{M}),$$

hence $\mathscr{M}$ is a so-called J-*positive* subspace. Then it is known (see [1, Chapter 1, Section 7]) that $\mathbf{C}^n$ is the (algebraic) direct sum of $\mathscr{M}$ and its J-*orthocomplement* $\mathscr{N}$, defined by

$$\mathscr{N} \equiv \{z; [y, z] = 0 \quad \forall y \in \mathscr{M}\},$$

both of which are invariant for A. By definition we have $\sigma(A|_{\mathscr{N}}) \subset (0, \infty)$. Now any vector $x \in \mathbf{C}^n$ is uniquely written as

$$x = y + z \quad \text{with } y \in \mathscr{M}, z \in \mathscr{N}$$

and for any $0 < \lambda < 1$

$$A\{\lambda A + (1 - \lambda)I\}^{-1}x = (A|_{\mathscr{N}})\{\lambda(A|_{\mathscr{N}}) + (1 - \lambda)I\}^{-1}z,$$

which guarantees the convergence of the integral

$$\frac{1}{\pi}\int_0^1 A\{\lambda A + (1 - \lambda)I\}^{-1}(\lambda(1 - \lambda))^{-\frac{1}{2}}x \,\mathrm{d}\lambda.$$

This completes a proof. $\square$

Theorem 6. *Let A, B be J-selfadjoint matrices with non-negative eigenvalues. If*

$$I \overset{J}{\geqslant} A \overset{J}{\geqslant} B,$$

then J-selfadjoint square roots $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ are well defined and

$$I \overset{J}{\geqslant} A^{\frac{1}{2}} \overset{J}{\geqslant} B^{\frac{1}{2}}.$$

Proof. When $\sigma(A), \sigma(B) \subset (0, \infty)$, this is a consequence of Theorem 4 with the operator monotone function $f(t) = t^{\frac{1}{2}}$ on $(0, \infty)$.

When A or B has 0 as its eigenvalue, the square roots are defined by the integral $(*)$ which are convergent by Lemma 5. As in the proof of Theorem 4 $JA \geqslant JB$ implies that

$$JA\{\lambda A + (1 - \lambda)I\}^{-1} \geqslant JB\{\lambda B + (1 - \lambda)I\}^{-1} \quad (0 < \lambda < 1).$$

Therefore we have $JA^{\frac{1}{2}} \geqslant JB^{\frac{1}{2}}$. This completes a proof. $\square$

Corollary 7

$$I \overset{J}{\geqslant} A^{\#}A \overset{J}{\geqslant} B^{\#}B \Longrightarrow I \overset{J}{\geqslant} (A^{\#}A)^{\frac{1}{2}} \overset{J}{\geqslant} (B^{\#}B)^{\frac{1}{2}}.$$

Proof. As mentioned in Section 1 the assumption implies that $\sigma(A^\# A),\ \sigma(B^\# B) \subset [0, \infty)$. Now apply Theorem 6 to complete a proof. $\square$

The square roots $(A^\# A)^{\frac{1}{2}}$ is called the *J-modulus* of A. Now A admits a *J-polar representations* $A = U(A^\# A)^{\frac{1}{2}}$ with J-unitary U, that is, $U^\# U = I$ (see [1, Chapter 4, Section 1]).

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